What is Majorana theory?

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In the classification of finite simple groups, it was shown that there are exactly 26 sporadic simple groups, groups which do not lie in any of the infinite families which make up the rest of the classification. The Monster simple group, $M$, is the largest of these sporadic groups and contains 20 of them as subgroups or quotients of subgroups. It has order

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53}.$$ 

The existence of the Monster was independently conjectured by B. Fischer and R. Griess in 1973. It was first constructed by Griess in [Gri82] as the automorphism group of $V_M$, a 196,884-dimensional commutative, non-associative, real algebra known as the Griess or Monster Algebra. The algebra $V_M$ is the direct sum of an irreducible module of $M$ and the one-dimensional trivial module. This construction was later simplified by J. H. Conway [Con84] and J. Tits [Tit84].

It is well known (see [CCN+85]) that $M$ contains two conjugacy classes of involutions which we will refer to as $2A$ and $2B$, where $2A$ is the smaller of the two. These two classes play an important role in the study of the Monster. The $2A$ involutions are 6-transpositions in that the product of any two has order at most 6. Moreover, $M$ is generated by the $2A$ involutions, making it a 6-transposition group.

Conway [Con84] showed that for each $x \in 2A$, we may define an idempotent vector $\psi(x) \in V_M$ called the 2A-axis corresponding to $x$. S. P. Norton and Conway [Con84, Nor96] have described all the subalgebras generated by two 2A-axes $\psi(x)$ and $\psi(y)$. They are known as dihedral subalgebras and are completely determined by the conjugacy class in $M$ of the product $xy$, for which there are nine distinct possibilities:

$$1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A.$$ 

The type of the dihedral subalgebra generated by $\psi(x)$ and $\psi(y)$ is defined to be the conjugacy class of the product $xy$.

Simply by its position as the largest of the sporadic groups, the Monster is an object of great significance in group theory. However, as mathematicians began to work on the so-called “Friendly Giant”, they began to see some surprising connections between the Monster and another area of mathematics, modular functions.

In particular, in 1978, John McKay noticed that the first degree coefficient in the $q$-expansion of
the elliptic modular function (commonly denoted $j$) differs from the dimension of the smallest irreducible representation of $\mathbb{M}$ by just one. As R.E. Borcherds writes in [Bor02], “it was clear to many people that this was just a meaningless coincidence; after all, if you have enough large integers from various areas of mathematics then a few are going to be close just by chance”.

However, this view was not universally shared and a few mathematicians, including Borcherds himself, believed that this “coincidence” may in fact be a clue to a deeper connection between the two objects. In [CN79], Conway and Norton proposed a number of conjectures formalising this link, which they christened “Monstrous Moonshine”.

In [FLM88], I. Frenkel, J. Lepowsky and A. Meurman constructed an infinite graded algebra $V^\natural$ known as the Monster Module such that $\text{Aut}(V^\natural) = \mathbb{M}$ and its graded dimension is the function $j$. Using this, in 1992 Borcherds [Bor92] proved Conway and Norton’s conjectures.

The module $V^\natural$ lies in a class of mathematical objects known as vertex operator algebras, or VOAs. Both VOAs and their close relations, vertex algebras, were first introduced as purely mathematical tools but have since been shown to have applications in certain areas of physics. They play a key role in the motivation behind Majorana Theory.

Now let $V = \bigoplus_{n=0}^{\infty} V_n$ be a real VOA such that $V_0 = \mathbb{R}1$ and $V_1 = 0$. Then the space $V_2$ has the structure of a commutative non-associative algebra and is referred to as the generalised Griess algebra of $V$. Crucially, when $V = V^\natural$, $V_2 = V_M$, which means that the Griess algebra is an example of a generalised Griess algebra.

M. Miyamoto [Miy96] showed that there exist involutions $\tau_a \in \text{Aut}(V)$ which correspond to special generators $a \in V_2$ known as Ising vectors. Moreover, when $V = V^\natural$, the vectors $\frac{1}{2}a$ are $2A$-axes and the $\tau_a$ are $2A$-involutions of $\mathbb{M}$. S. Sakuma, a student of Miyamoto, then proved the following result, which we refer to later in the text as Sakuma’s theorem.

**Theorem 0.1.** If $V_2$ is a generalised Griess algebra and $a_1, a_2 \in V_2$ are Ising vectors then the subalgebra $\langle a_1, a_2 \rangle$ is isomorphic to one of the nine dihedral subalgebras of the Monster algebra.

This was a remarkable result which reproved the classification of the dihedral algebras of the Griess Algebra, but in the more general setting of VOAs. If offered hope that this approach might offer a new way of studying the Monster. However, VOAs are very complex objects whose roots lie in theoretical physics. This made it very hard for mathematicians to fully exploit their potential in studying the Monster.

Majorana theory offers a solution to this problem. In 2009, A.A. Ivanov [Iva09] introduced Majorana theory as an axiomatisation of certain properties of generalised Griess algebras which reframed the approach of VOAs in a way which could easily be applied to group theoretic problems.

In particular, the Majorana axioms allow the construction of non-associative real algebras known as Majorana algebras, which can be considered as analogues of generalised Griess algebras. Majorana algebras are generated by certain idempotents, known as Majorana axes, which correspond to involutions in the automorphism group of the algebra, know as Majorana involutions. Major-
Majorana axes and involutions correspond to Ising vectors and Miyamoto involutions respectively.

In 2010, [IPSS10] Ivanov et al proved that a Majorana algebra generated by two Majorana axes is isomorphic to one of the dihedral subalgebras of $V_{55}$. This reproved Sakuma’s theorem using the Majorana axioms, demonstrating the potential of the new theory.

Since then, a number of publications have further developed the theory (cf. [CRI14], [Dec14], [IPSS10], [Iva11b], [Iva11a], [IS12a], [IS12b], [Ser12]). In particular, Majorana theory has been used to construct a number of important subalgebras of the Griess algebras including two algebras of dimension 20 and 26 corresponding to the 2A-generated $A_5$-subgroups in the Monster [IS12a].

Majorana algebras show a remarkable tendency to embed into the Griess algebra, with only a few of the known Majorana algebras existing independently. In fact, A. A. Ivanov has posed the following Straight Flush Conjecture.

**Conjecture 0.2.** Every indecomposable Majorana algebra in which 2, 3, 4, 5 and 6 appear as the order of the product of two Majorana involutions, always embeds into the Griess algebra.

If the conjecture was true, it would place the Griess algebra as the universal object in the class of Majorana algebras, adding weight to the belief that Majorana theory will prove to be a crucial tool in the study of the Monster and the Griess algebra.

Finally, we note that a single Ising vector generates a VOA known as the Virasoro VOA $L(\frac{1}{2}, 0)$ of central charge $\frac{1}{2}$. This algebra is isomorphic to the operator algebra of the two-dimensional Ising model, which is equivalent to the lattice model of free Majorana fermions, which are defined to be fermions which are their own antiparticle (see [GNT98, pp. 101-104]). It is this link which lends Majorana theory its name.

**References**


